A new finite element with transverse shear deformations included for shell strength analysis

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Using the Mindlin-Reissner hypothesis about the law of displacement distribution directly from the equations of the three dimensional theory of elasticity is received the full shell energy functional in arbitrary curvilinear system of co-ordinates. From the stationary condition of the functional a new model four-node shell for the moderately thick shell analysis is constructed. The final expressions for stiffness matrix and vector of the external forces are brought. The proposed element is free from the locking effect.

As against existing numerous isoparametric finite elements (FE), in the present article isoparametric co-ordinates are entered at the last stage of reception of final expressions for the stiffness matrix and the vector of external forces of the finite element and can be considered only as a convenient means for numerical integration.

The account of transverse shear deformations allows essentially to expand area of application received shell FE for stress-strain analysis of moderately thick shells. The requirement C^1 -the continuity of basic functions for displacements according to the Kirchoff theory is replaced by the requirement C^0. The last continuity requirement allows to construct more simple models of finite elements. However at the account of the shear strain energy there are the specific difficulties. The direct account of the shear strain energy leads to the appear of the locking effect when the shell thickness is reduced. The mentioned defect is displayed as a consequence of inevitable limited accuracy of numerical calculations. In result the use of a such FE on the stress-strain analysis shift FE for rather thin shells not only does not improve the result account on the Kirchoff theory, but can bring a complete distortion of the valid stress-strain picture.

To construction of finite elements for plates and shells, free from the specified above lack, enough many works [1-13] are devoted.

Below, in a combination with the new approach [14] for construction of isoparametric finite elements, a simple effective method [5] was used for construction of shell finite element with shear effects included. The locking effect is excluded.

1. Model of a thin shell

Consider a shell of constant thickness limited by two parallel surfaces. Let $2h$ is the thickness of shell. Connect with middle shell surface the Gauss co-ordinate lines x^λ, $\lambda=1,2$. The third axis x^3 is directed on the normal to middle surface of shell. By $a_{\alpha\beta}$ and $b_{\alpha\beta}$ we denote accordingly the coefficients of the first and second square forms of middle surface of the shell ($\alpha,\beta = 1,2$). For enough thin shell it is possible to assume that

$$\left|hb_\lambda^\mu\right| << 1 \ , \tag{1}$$

The last inequality allows to neglect by the distinction in magnitudes of metric coefficients $a_{\alpha\beta}$, $b_{\alpha\beta}$ for middle surface and for any parallel surface. In accepted system of co-ordinates the elements of metric tensor g^{ij} are equal to

$$g^{\lambda\mu} = a^{\lambda\mu} \ , \ g^{\lambda 3} = 0 \ , \ g^{33} = 1 \ . \tag{2}$$

Notice that the Greek indexes accept 1 or 2 and the Latin indexes - from 1 up to 3

2. Additional hypothesises

The full energy of deformation F of an any elastic body consists of potential energy П and work of external forces A:

$$F = P - A \tag{3}$$

The potential energy of deformation is equal to

$$P = \frac{1}{2}\iiint_V e^{ij,mn} e_{ij} e_{mn} dV, \tag{4}$$

where $e^{ij,mn}$ - a tensor of elastic constants. For isotropic material

$$e^{ij,mn} = l\, g^{ij} g^{mn} + m(g^{im} g^{jn} + g^{in} g^{jm})$$

Here l and mare the Lame coefficients describing elastic properties of the material.

The linear tensor of deformation, included in these expressions, is calculated by formula

$$e_{mn} = \frac{1}{2}(u_{m;n} + u_{n;m})$$

where ";" - designates three dimensional covariant differentiation.

The expression for work of external forces can be written in the form

$$A = \iiint_V g^n u_n dV, \tag{5}$$

where g^n - components of the external forces per unit of volume.

As additional hypothesises we shall accept the Mindlin- Reissner hypothesises:

$$e_{33} = 0; \quad e_{l\ 3;3} = 0. \tag{6}$$

In chosen system of curvilinear co-ordinates we have the following equality:

$$e^{ab,gs} e_{ab} e_{gs} = e^{ab,gs} u_{a;b} u_{g;s}.$$

In result the expression for potential energy of deformation is simplified and can be written down as

$$P = \frac{1}{2}\iiint_V [e^{ab,gs} u_{a;b} u_{g;s} + 4ma^{ag} e_{a3} e_{g3}]\sqrt{a}\, dx^1 dx^2 dx^3, \tag{7}$$

where $e^{ab,gs} = la^{ab} a^{gs} + m(a^{as} a^{bg} + a^{ag} a^{bs})$.

3. Functional of energy

The equations (5) and (7) are fair in chosen co-ordinate system under restriction (1) for any three dimensional elastic isotropic body. For transition from the equations of the three dimensional theory of elasticity to two dimensional shell equations it is necessary the expressions for work of external forces (5) and potential energy of deformation (7) integrate on shell thickness. But previously express the displacements and strains at any point through the appropriate parameters of middle shell surface. We expand these quantities into the Tailor series:

$$u_\lambda(x^3) = u_\lambda\Big|_{(x^3=0)} + x^3 u_{\lambda;3}\Big|_{(x^3=0)} + \frac{1}{2}(x^3)^2 u_{\lambda;3;3}\Big|_{(x^3=0)} + \ldots$$

$$u_3(x^3) = u_3\Big|_{(x^3=0)} + x^3 u_{3;3}\Big|_{(x^3=0)} + \frac{1}{2}(x^3)^2 u_{3;3;3}\Big|_{(x^3=0)} + \ldots \quad (8)$$

$$\varepsilon_{\lambda 3}(x^3) = \varepsilon_{\lambda 3}\Big|_{(x^3=0)} + x^3 \varepsilon_{\lambda 3;3}\Big|_{(x^3=0)} + \ldots.$$

The expressions (8) do not allow to solve a problem of reduction three dimensional equations of the theory of elasticity to two dimensional equations of shell theory as they contain derivatives on normal co-ordinate x^3.

We shall enter, as additional unknown parameters, the rotations of normal to the middle surface θ_λ, $\lambda = 1,2$:

$$\theta_\lambda = u_{\lambda;3} = 2\varepsilon_{\lambda 3} - u_{3;\lambda}\,, \quad (9)$$

In result we receive the following approximate expressions for displacements and strains:

$$u_\lambda(x^3) = u_\lambda\Big|_{(x^3=0)} + x^3\theta_\lambda\Big|_{(x^3=0)}$$

$$u_3(x^3) = u_3\Big|_{(x^3=0)} \quad (10)$$

$$\varepsilon_{\lambda 3}(x^3) = \varepsilon_{\lambda 3}\Big|_{(x^3=0)}$$

The obtained expressions allow to get the shell equations from the equations of the theory of elasticity.

By substituting (10) in (7), for potential energy of deformation we receive

$$\Pi = \frac{1}{2}\iiint_V [e^{\alpha\beta,\gamma\sigma}(u_\alpha + x^3\theta_\alpha)_{;\beta}(u_\gamma + x^3\theta_\gamma)_{;\sigma} + 4\mu a^{\alpha\gamma}\varepsilon_{\alpha 3}\varepsilon_{\gamma 3}]\sqrt{a}\,dx^1dx^2dx^3 .$$

In the last expression we can pass from space covariant differentiation to two dimensional covariant differentiation on the middle surface using the equality:

$$v_{\mu;\lambda} = \nabla_\lambda v_\mu - b_{\mu\lambda} v_3$$

where by ∇_λ we shall designate the symbol of two dimensional covariant differentiation.

Integration of the received expression on thickness from -h up to +h gives

$$\Pi = \iint_\Omega [e^{\alpha\beta,\gamma\sigma}\{h\nabla_\alpha u_\beta \nabla_\gamma u_\sigma - 2hb_{\alpha\beta}u_3\nabla_\gamma u_\sigma + hb_{\alpha\beta}b_{\gamma\sigma}u_3u_3 + \frac{1}{3}h^3\nabla_\alpha\theta_\beta\nabla_\gamma\theta_\sigma\} + 4\mu h a^{\alpha\gamma}\varepsilon_{\alpha 3}\varepsilon_{\gamma 3}]\sqrt{a}\,dx^1dx^2 , \quad (11)$$

where $\varepsilon_{\lambda 3} = \frac{1}{2}(\theta_\lambda + \nabla_\lambda u_3)$.

Similarly for the work of external forces, when the external forces are put statically, we may receive the expression

$$A = \iint_\Omega [a^{\alpha\beta}(\{p_\alpha\}u_\beta - h[p_\alpha]\theta_\beta) + \{p_3\}u_3]\sqrt{a}\,dx^1dx^2 . \quad (12)$$

Here through

$$\{p_m\} = p_m^{(+)} + p_m^{(-)} + \int_{-h}^{h} g_m dx^3 \qquad [p_m] = p_m^{(+)} - p_m^{(-)} + \frac{1}{h}\int_{-h}^{h} x^3 g_m dx^3$$

the external forces applied to middle surface are denoted; $p_m^{(+)}$ And $p_m^{(-)}$ are components of the surface forces on upper and lower surfases of the shell correspondly; g_m are components of a volume forces vector

The received expressions for potential energy of deformation (11) and work of external forces (12) can directly be used in finite-element procedures. However for thin shells the received system of linear algebraic equations will possess with the locking effect. In order to avoid the locking effect it is necessary to transform the expression for potential energy of deformation.

4. Transformation of transverse shear strain

The analysis of the obtained equations allows to reach the conclusion that for construction "qualitative" finite element with including of transverse shear deformations, it is necessary to express shear strains e_{l3} through the rotations Q_l. Under "qualitative" finite element we assume such element by means of which the influence of shear deformations can be correctly estimated independently on shell thickness.

The variation of transverse shear energy can be written in the form:

$$dP(e_{s3}) = \iint\limits_G [e^{ab,gs} \frac{4}{3} h^3 \nabla_a Q_b \nabla_g de_{s3} + 8mha^{bs} e_{b3} de_{s3}] \sqrt{a}\, dx^1 dx^2 \ .$$

Taking into account the identity $\nabla_l A \nabla_m B = \nabla_l (A \nabla_m B) - A \nabla_l \nabla_m B$, the variation dP can be transformed into expression:

$$dP(e_{s3}) = \iint\limits_G e^{ab,gs} \frac{4}{3} h^3 \nabla_g (de_{s3} \nabla_a Q_b) \sqrt{a}\, dx^1 dx^2 +$$

$$+ \iint\limits_G [e^{ab,gs} \frac{4}{3} h^3 de_{s3} \nabla_a \nabla_g Q_b + 8mha^{bs} e_{b3} de_{s3}] \sqrt{a}\, dx^1 dx^2$$

At covariant differentiation the components $e^{\alpha\beta,\gamma\sigma}$ for homogeneous material behave as constant values. It permits to transform the first integral in the last expression by means of the Gauss-Ostrogradsky formula

$$\iint\limits_G \nabla_l A^l \, dG = \oint\limits_L A^l n_l \, dL$$

to the expression

$$dP = \oint\limits_L e^{ab,gs} \frac{4}{3} h^3 de_{s3} \nabla_a Q_b n_g dL +$$

$$+ \iint\limits_G [-e^{ab,gs} \frac{4}{3} h^3 \nabla_a \nabla_g Q_b + 8mha^{bs} e_{b3}] de_{s3} \sqrt{a}\, dx^1 dx^2$$

Here n_l are components of unit normal to shell contour L.

Minimisation of potential energy with respect to shear deformations leads to some equations of equilibrium

$$e^{ab,gs} \frac{4}{3} h^3 \nabla_a \nabla_g Q_b - 8mha^{bs} e_{b3} = 0, \ s = 1,2. \qquad (13)$$

From (13) the shear deformations can be expressed through the rotations Q_b:

$$e_{l\,3} = \frac{h^2}{6m} a_{la}\, e^{ab,gs} \nabla_b \nabla_g Q_s \; . \tag{14}$$

After substitution (14) in (11) and after some replacement of indexes of summation, the following expression for potential energy of deformation in the metric of middle surface can be received:

$$\begin{aligned} P = \iint_G & [e^{ab,gs}\{h\nabla_a u_b \nabla_g u_s - 2hb_{ab} u_3 \nabla_g u_s + hb_{ab} b_{gs} u_3 u_3 + \\ & + \frac{1}{3} h^3 \nabla_a \P_b u_3 \nabla_g \P_s u_3 + \frac{1}{9m} h^5 a_{al}\, e^{lm,nh} \nabla_b \nabla_m \nabla_n Q_h \nabla_g \P_s u_3\} + \\ & + \frac{h^5}{9(m)^2} a_{ml}\, e^{mn,ab} e^{lh,gs} \nabla_n \nabla_a Q_b \nabla_h \nabla_g Q_s]\sqrt{a}\, dx^1 dx^2 \end{aligned} \tag{15}$$

Similarly, for calculation of the work of external forces we receive

$$A = \iint_G [a^{ab} (\{p_a\} u_b + h[p_a](\P_b u_3 - \frac{h^2}{6m} a_{bl}\, e^{ml,dh} \nabla_m \nabla_d Q_h)) + \{p_3\} u_3]\sqrt{a}\, dx^1 dx^2 \tag{16}$$

Directly from consideration of expressions (15) and (16), it is possible to show that at $h \rightarrow 0$ the members connected with shear deformations also aspire to zero and (15) and (16) pass in the appropriate expressions for the Kirchhoff theory. Thus the expression (15) and (16) can directly be used for construction "qualitative" isoparametric finite element with shear deformations included.

5. Method of finite elements

Let's assume

$$u_l = \tilde{u}_l^{(b)} j_{(b)}; \quad u_3 = \tilde{u}_3^{(b)} y_{(b)}; \quad Q_l = \tilde{Q}_l^{(b)} y_{(b)}, \quad (l = 1,2) \tag{17}$$

where b = 1, 2,..., n - the numeration of nodes of finite element net; $j_{(b)}$, $y_{(b)}$ - given basic functions determined in the vicinity of node (b); $\tilde{u}_l^{(b)}, \tilde{u}_3^{(b)}, \tilde{Q}_l^{(b)}$ are unknown constants.

In a general case the form of e - vicinity can be arbitrary. For the quadrangle the basic functions, ensuring conditions of convergence, look like

$$j_{(b)} = (1 - |h^1|)(1 - |h^2|); \quad y_{(b)} = (1 - |h^1|)^2 (1 + 2|h^1|)(1 - |h^2|)^2 (1 + 2|h^2|),$$

where $|\eta^1| \le 1, |\eta^2| \le 1$ are local co-ordinates inside a rectangular e - vicinity of the node (b).

The substitution of expressions (17) in (15) and (16) transforms the energy functional F into quadratic function of unknown values $\tilde{u}_\lambda^{(b)}, \tilde{u}_3^{(b)}, \tilde{\Theta}_\lambda^{(b)}$. The condition of existence of a minimum of this function

$$\frac{\P F}{\P \tilde{u}_l^{(c)}} = 0; \quad \frac{\P F}{\P \tilde{u}_3^{(c)}} = 0; \quad \frac{\P F}{\P \tilde{Q}_l^{(c)}} = 0.$$

leads to the system of linear algebraic equations for determination of unknown parameters $\tilde{u}_{l}^{(b)}, \tilde{u}_{3}^{(b)}, \tilde{Q}_{l}^{(b)}$:

$$\begin{Vmatrix} k_{(b),(c)}^{lm} & k_{(b),(c)}^{l3} & 0 \\ k_{(b),(c)}^{3l} & k_{(b),(c)}^{33} & r_{(b),(c)}^{3l} \\ 0 & r_{(b),(c)}^{l3} & r_{(b),(c)}^{lm} \end{Vmatrix} \begin{Bmatrix} \tilde{u}_{l}^{(b)} \\ \tilde{u}_{3}^{(b)} \\ \tilde{Q}_{l}^{(b)} \end{Bmatrix} = \begin{Bmatrix} p_{(c)}^{m} \\ p_{(c)}^{3} \\ g_{(c)}^{m} \end{Bmatrix}, \quad (b, c = 1, M) \qquad (18)$$

Here the summation is carried out on all repeating top and bottom indexes, M - number of net nodes.

In conclusion we write the formulas for calculation of elements of stiffness matrix and external load vector:

$$k_{(b),(c)}^{lm} = 2\iint_{G} e^{ab,gs} h \nabla_{ab}^{l} j_{(b)} \nabla_{gs}^{m} j_{(c)} \sqrt{a}\, dx^1 dx^2$$

$$k_{(b),(c)}^{l3} = -2\iint_{G} e^{ab,gs} h b_{ab} y_{(b)} \nabla_{gs}^{l} j_{(c)} \sqrt{a}\, dx^1 dx^2$$

$$k_{(b),(c)}^{33} = 2\iint_{G} e^{ab,gs} [h b_{ab} y_{(b)} b_{gs} y_{(c)} + \frac{1}{3} h^3 D_{ab} y_{(b)} D_{gs} y_{(c)}] \sqrt{a}\, dx^1 dx^2$$

$$r_{(b),(c)}^{3l} = 2\iint_{G} e^{ab,gs} \frac{1}{9m} h^5 a_{ap} e^{pr,nh} \nabla_b \nabla_r \nabla_{nh}^{l} y_{(b)} D_{gs} y_{(c)} \sqrt{a}\, dx^1 dx^2 \qquad (19)$$

$$r_{(b),(c)}^{lm} = 2\iint_{G} \frac{h^5}{9(m)^2} a_{pr} e^{pn,ab} e^{rh,gs} \nabla_n \nabla_{ab}^{l} y_{(b)} \nabla_h \nabla_{gs}^{m} y_{(c)} \sqrt{a}\, dx^1 dx^2$$

$$p_{(c)}^{m} = \iint_{G} a^{ma} \{p_a\} j_{(c)} \sqrt{a}\, dx^1 dx^2$$

$$p_{(c)}^{3} = \iint_{G} (a^{ab} h [p_a] ¶_b y_{(c)} + \{p_3\} y_{(c)}) \sqrt{a}\, dx^1 dx^2 \qquad (20)$$

$$g_{(c)}^{m} = -\iint_{G} a^{ab} [p_a] \frac{h^3}{6m} a_{bl} e^{lp,dh} \nabla_p \nabla_{dh}^{m} y_{(c)} \sqrt{a}\, dx^1 dx^2$$

The middle surface metric enters directly in the tensor of elastic constants.
At record of the formulas (19) and (20) following system of the differential operators was entered

$$\nabla_{ab}^{l} u_l = d_a^l ¶_b u_l - G_{ab}^{l} u_l \qquad D_{ab} u_3 = ¶_a ¶_b u_3 - G_{ab}^{l} ¶_l u_3$$

where $\delta_{\alpha}^{\lambda}$ are the Kronecer symbols.

Given above dependencies for the stiffness matrix and the load vector can be used at construction of isoparametric finite elements for shells with arbitrary co-ordinate net and also environments of variable thickness. By means of respective alteration of tensor of elastic constants these dependencies can be used for orthotropic shells.

It is important to note once more, that received finite element allows not only to take into account influence of shear deformations, but also is suitable for account of as is wished thin shells.

References

1. Boldichev V.V. Double approximation of rotations at account of plates of average thickness by a method of finite elements. / News VNIIT named by B.E. Vedeneev, Vol. 133, 1979. p. 68-74.
2. Boldichev V.V. About connection of the various variants of finite element method at the decision of degenerating problems / Method of finite elements and account of structures. Transaction of Leningrad politechnical institute, N 405, 1985, p. 36-41.
3. Djachkin A.A. Development of a finite element approach to accounts of thin-walled designs with shear deformation included. / Theses of the dissertation for candidate of science degree. Moscow, MGTU named by N.E. Bauman, 1993.
4. Kramareva I.V., Pautov A.N. Regularization of stiffness matrix in finite element analysis of beams and plates with shear deformation included. //Applied Problems of Strength and Plasticity / High School Collection, 1986, p. 66-72 (Gorkovsky State University).
5. Postnov V.A., Slezina N.G. The account of physical and geometrical nonlinearity by use of finite element method in problems of shells of rotation .//Trans. AN USSR, MTT, 1979, p.78-85.
6. Saharov A.S., Kislovsky V.N., Kirichevsky V.V., Altenbah I. Method of finite elements in the mechanic of firm body./ Kiev, Vicha shkola, 1982, p. 680 .
7. J.L. Batoz, P.Kardeur. A discrete shear triangular nine D.O.F. Element for the analysis of thick to very thin plates. Int. J. numer. Meth. Eng., vol. 28, 1989, p. 533-560.
8. Gellert M. A new method for derivation of locking -finite element via mixed hybrid formulation // Jnt. J.Numer. Meth. Eng., vol. 26, 1988, p. 1185-1200.
9. Kamoulakos A. Understanding and improving the reduced integration of Mindlin shell elements // Int. J. numer. Meth. Eng., vol. 26, 1988, p. 2009-2029.
10. Kotili I. A new discrete Kirchhoff-Mindlin element based on Mindlin-Reissner plate theory and assumed shear strain fields // Int. J. numer. Meth. Eng., vol. 36, 1993, p. 1859-1908.
11. Paradopoulos P., Taylor R.L. A triangular element based on Reissner-Mindlin plate theory // Int. J. numer. Meth. Eng., vol.30, 1990, p.1029-1049.
12. Stolarski H.K., Chiang M.Y.M. Thin plate elements with relaxed Kirchhoff constraints. //Int. J. numer. Meth. Eng., vol.26, 1988, p.913-933.
13. Zienkiewicz O.C., Lefebvre D. A robust triangular plate bending element of the Reisner-Mindling type // Int. J. numer. Meth. Eng., vol.26, 1988, p.1169-1184.
14. Postnov V.A., Trubachev M.I. New model of isoparametric finite element for shell stress analysis. // Trans. AN USSR, MTT, 1994, p .78-85