Inverse Eigenvalue Problem and Identification of Elastic Systems

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The method of determination of boundary conditions of a beam by mathematical processing of experimentally measured frequent spectra of considered elastic system (beam) and some systems received from initial system as " a black box " by entering of certain changes of its boundary conditions is proposed. By numerical experiment with using of the generator of random numbers the stability of offered algorithm for solution of the considered problem has been demonstrated.

1. Introduction

The decision of a problem of identification of elastic mechanical systems by the analysis of their frequent spectra, received from experiment, has the extremely important practical meaning. Developed to the present of time the approaches of the decision of the given problem [1-5] are based on use of properties of oscillation of matrixes [6,7]. The work [4] specifies a way of the decision of a problem of identification any beam system.

The decision of a problem of identification of one dimensional of elastic systems is appreciably simplified, if the systems given about frequent spectra are known at some simple boundary conditions. To create such conditions of fastening by experiment it is frequently practically impossible. Such spectra can be received by a numerical calculation, but for this the complete information on all parameters, determining boundary conditions of considered elastic system is required. To the decision of this private problem of identification the present work is devoted.

2. Determination of concentrated masses

Let dynamic properties of considered system (the beam with variable rigidity and mass, with arbitrary boundary conditions) approximately can be described by a diagonal mass matrix M and matrix of rigidity K:

$$M = M^T = [\ m_i\ ], \qquad K = K^T = [\ k_{ij}\ ]$$

At such approach the initial beam is replaced by a weightless beam with weights concentrated in some beam sections (points) along its length. The inertia of linear moving of weights as well the inertia of rotation are taken into account. Thus in each nodal point of the beam we have two generalised coordinates for determination of linear and angular moving. The definitely-positive matrixes M and K have the n-th order. The free vibration of considered elastic system are described by the matrix equation:

$$M\ddot{q}(t) + Kq(t) = 0 \tag{1}$$

The equation for determination of a square of own frequencies λ_i ($i = 1, \ldots, n$) :

$$P_n(\lambda) = |K - \lambda^2 M| = 0 \tag{2}$$

After expanding the determinant $P_n(\lambda)$ on elements of the s-th row of the matrix $|K-\lambda^2 M|$, we obtain

$$P_n(\lambda) = (k_{ss} - m_s\lambda^2)P^{(s)}_{n-1} + \sum_{i=1}^{n-1} k_{si}(1-\delta_{si})P^{(i)}_{n-1} = \alpha \prod_{i=1}^{n}(\lambda_i^2 - \lambda^2) \tag{3}$$

Here $R_{n-1}^{(s)}(l\ )$- algebraic addition for the i-th element of the row; l_i - natural frequency of considered elastic system; d_{si} - Kronecer symbol;

$$a = (-1)^n \prod_{i=1}^{n} m_i \qquad (4)$$

Now attach to the system additional mass m^s and stiffness k^s for s-th general coordinate. Then for the modified elastic system by analogy with (3) the expression representing the left part of frequency equation will be written down as

$$\overline{P}_n(l\ ) = (k_{ss} - m_s l^2 + k^s - m^s l^2)P_{n-1}^{(s)} + \sum_{i=1}^{n-1} k_{si}(1 - d_{si})P_{n-1}^{(i)} = b \prod_{i=1}^{n}(m_i^2 - l^2) \qquad (5)$$

were m_i -the natural frequency of modified elastic system:

$$b = (-1)^n \prod_{i=1}^{n}(m_i + m^s d_{in}) \qquad (6)$$

Subtracting expression (3) from (5), we shall receive

$$(k^s - m^s l^2)P_{n-1}^{(s)}(l\ ) = b \prod_{i=1}^{n}(m_i^2 - l^2) - a \prod_{i=1}^{n}(l_i^2 - l^2) \qquad (7)$$

Substitution $l^2 = d_s^2 = k^s / m^s$ into (7) gives

$$b = a \prod_{i=1}^{n} \frac{(l_i^2 - d_s^2)}{(m_i^2 - d_s^2)} \qquad (8)$$

Introducing a and b, determined by formulas (4) and (6), in expression (8), we receive the formula for determination of weight m_s considered elastic system on a direction of generalised coordinate q_s:

$$m_s = \frac{m^s}{A_s - d_s^2}, \quad A_s = \prod_{i=1}^{n} \frac{l_i^2 - d_s^2}{m_i^2 - d_s^2} \qquad (9)$$

Knowing the experimental values of frequencies of initial system l_φ and also frequencies m_j^s for system received from the initial by entering of additional rigidity k^s and weight m^s, with the help of dependencies (9) it is possible to define weight m^s.

3. Frequency equations of changed elastic systems

Take as generalised coordinates for the modified system obtained from the given system by attaching of additional weight m_s and rigidity k_s the principal coordinates $p_j(t)$ of the considered system. Then the equations of free vibration of modified system will be written in the form:

$$M_j\ddot{p}_j + N_j p_j + m^s n_{sj}\sum_{r=1}^{n} n_{sr}\ddot{p}_r + k^s n_{sj}\sum_{r=1}^{n} n_{sr}p_r = 0\ , \quad j = \overline{1,n}$$

which by substitution $p_j(t) = A_j\cos(l\,t + e_j)$ will be transformed to system of the equations:

$$A_j M_j(l_j^2 - l^2) - m n_{sj} l^2 B + k^s n_{sj} B = 0, \qquad (j = \overline{1,n}) \qquad (10)$$

where

$$B = \sum_{r=1}^{n} n_{sr} A_r \qquad (11)$$

Excepting from (11) with the help (10) the value A_i, we shall find the frequency equation for the system, received from initial elastic system by introduction of additional weight m^s and rigidity k^s on the direction of generalised coordinate q_s:

$$\frac{1}{m^s\left(l^2 - \frac{k^s}{m^s}\right)} = \sum_{r=1}^{n} \frac{n_{sr}^2}{M_r(l_r^2 - l^2)} \qquad (12)$$

Directly from (12) if to put $k^s = \infty$, we shall receive the frequent equation of system under condition of $q_s=0$:

$$\sum_{r=1}^{n} \frac{n_{sr}^2}{M_r(l_r^2 - l^2)} = 0 \qquad (13)$$

It is possible to show, that if m_i (i=1,2,.. N-1) are roots of the equation (13), the relation n^2_{si} / M_i is defined by the formula

$$\frac{n_{si}^2}{M_i} = \frac{\prod_{j=1}^{j=n-1} (m_j^2 - l_i^2)}{\prod_{j=1}^{j=n} (l_j^2 - l_i^2 - d_{ij})} \qquad (14)$$

4. Determination of rigidity of an elastic support on a direction of s-th generalised coordinate

Probably from the equation (12) it is easier in comparison with the formula (14) to receive the ratio $b_i^s = n^2_{si} / M_i$:

$$\sum_{j=1}^{j=n} b_i^s c_{ij} = d_i^s \qquad \text{(i=1,2,..,n)} \qquad (15)$$

where

$$c_{ij} = \frac{1}{\prod_{k=1}^{k=n} (l_k^2 - m_i^2)} \prod_{k=1}^{k=n} \left[(l_k^2 - m_i^2)(1 - d_{kj}) + d_{kj}\right], \quad d_i = \frac{1}{m^s\left(m_i^2 - \frac{k^s}{m^s}\right)}, \qquad (16)$$

m_i - the frequency of the modified system by introduction of additional rigidity k^s and weight m^s, l_i - frequency of the initial system.

Let now the additional rigidity k_{suc}^s is connected to the basic rigidity K_s successively. Such connection is equivalent to parallel connection of the rigidity $k_{par}^s = -K_s^2 \big/ \left(K_s + k_{suc}^s\right)$. Then with reference to a considered(examined) case the frequent equation (12) can be transformed to a kind:

$$\sum_{j=1}^{n} \frac{b_j}{(\lambda_j^2 - \lambda^2)} = \frac{1}{m^s \lambda^2 + \dfrac{\mathrm{K}_s^2}{\mathrm{K}_s + k_{suc}^s}} \tag{17}$$

From here we receive the required formula for determination of rigidity K_s of the considered system:

$$K_s = \frac{\mathrm{B}}{2} + \sqrt{\left(\frac{\mathrm{B}}{2}\right)^2 + \mathrm{B} k_{suc}^s}\ , \tag{18}$$

$$\mathrm{B} = \frac{1}{\mathrm{A}} - m^s \nu_j^2\ , \quad A = \sum_{r=1}^{n} \frac{b_i^s}{(\lambda_r^2 - \nu_j^2)} \tag{19}$$

where ν_φ - the frequency of the system received from the initial system by introduction of successively additional weight m^s and successively connected rigidity k^s_{suc}.

5. Numerical example

We shall assume that the considered beam with variable weight and bending rigidity approximately can be replaced by discrete model with eight degrees of freedom (see fig. 1). In nodal points the concentrated weights are located: m_2=1, m_4=1, m_6=1.5, m_8=3, the moment of inertia of which are accordingly equal: m_1=0.5, m_3=1.5, m_5=2.5, m_7=3.5. Lengths of beam spaces are equal to: a_1=a_2=a_3=1; the bending rigidity in each space is accepted to be constant and the rigidity of elastic supports equal to K_1=6, K_2=5, K_7=4, K_8=10. The angular displacements of nodal points are denoted by q_1, q_3, q_5, q_7, and linear - through q_2, q_4, q_6, q_8 accordingly.

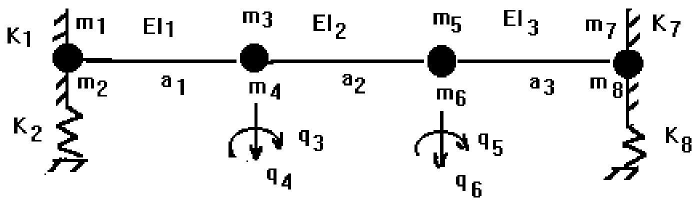

Fig.1. Model of a beam

The account of inertia of rotation of weights is undertaken in order to precise the discrete model of a beam. Usually at modelling of beams with variable weight only transverse forces of inertia are taken into account.

For mentioned above the beam model its own frequencies and also frequencies of some systems received by attaching to the considered beam the additional weights m^s and rigidity k^s were obtained from the experiment. The numerical values of these frequency spectrums are given in table 1. There also the corresponding values of additional attached weights and rigidities are indicated. It is supposed that all frequency spectrums, shown in table 1, can be received from experiment. Just it assumes the offered method.

Table 1. Natural frequency

i	λ_i	μ_i^7	μ_i^8	ν_i^7	ν_i^8
1	1.15	1.10	1.15	0.504	0.389
2	2.08	2.07	1.58	1.964	1.16
3	2.58	2.15	2.48	2.095	2.47
4	3.51	3.32	3.45	3.306	3.44
5	11.22	11.21	11.1	11.2	11.13
6	23.17	22.85	20.3	22.8	20.17
7	39.81	39.2	36.5	39.2	36.45
8	65.37	64.9	63.3	64.9	63.28
k^s	-	k_{par}^7=5	k_{par}^8=5	k_{suc}^7=5	k_{suc}^8=5
m^s	-	m^7=5	m^8=5	m^7=5	m^8=5

We shall show now that the information contained in tab. 1 is quite enough for definition of parameters describing the boundary conditions of the beam as a black box.

Introducing the frequencies l_i and m_i^7 in formula (9), we shall define the moment of inertia of weight m_7. Similarly, by the same formula (9) with using of frequencies l_i and m_i^8, the value of weight m_8 is defined.

From the solution of equations (15) the values b_i^7 and b_i^8 are defined. We shall notice that for determination b_i^7 it is necessary in dependence (16) to introduce m_i^2, and for definition b_i^8 it is necessary to use m_i^3. Further, by formulas (19) and (18) the required rigidities K_7 and K_8 can be calculated. We shall notice, that irrespective of accepted frequency n_i, we shall receive the same numerical values for the rigidities. In result of calculations the exact numerical values of all required quantities were received: m_7=3.5, m_8=3, K_7=10, K_8=4. Such result is a consequence that for their determination the exact numerical values of frequencies were used.

6. Numerical stability of settlement algorithm

Numerical stability of settlement algorithm, i.e. the estimation of influence of inevitable errors in experimentally measured frequencies on the result of our numerical calculation was considered. Numerical experiment carried out with use of the generator of random numbers for the various laws of distribution of frequency error the numerical experiment was fulfilled. The accounts have shown the following: the magnitude of the error of parameters, describing beam boundary conditions has the same order as the error of the initial information on experimentally measured frequencies. From here it is possible to come to the conclusion that stability of offered algorithm is enough satisfactory.

7. Final remark

All published work on inverse eigenvalue problem are limited by consideration only one-dimensional elastic systems. The decision of inverse problem for two-three dimensional elastic bodies will require the development of completely other approaches as the matrixes of rigidity of these elastic systems do not have oscillating properties.

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References

1. Gladwell G.M.L. Inverse problem in vibration. Martinis Nighoff Publishers, 1986. 340p.
2. Ram Yitshak M. Jnverse eigenvalue problem for a modified vibrating system. SIAM J. of applied Math., 1993,vol. 53, N 6.P. 1762-1775.
3. Postnov V.A. Identification of elastic systems by mathematical processing of frequency spectra, received from experiment. // Thesis of a All-Russia symposium " Dynamic and technological problems of the mechanics of continuous media". Moscow State Aviation Institute, Moscow, 1997.
4. Postnov V.A. Inverse eigenvalue problem for own dimensional with several branches elastic systems. // Applied problems of Strength and Plasticity. High Sch. Col., Nizhny Novgorod State University, 1997, N 56. In print.
5. Postnov V.A. Inverse eigenvalue problem for one dimensional elastic system. // Modern problems of mechanics of deformed bodies. Siktifkar, 1997.
6. Gantmaher F.R.Theory of matrixes.(in Russian). -M., Nauka, 1967.
7. Gantmaher F.R., Krein M.G. Oscillatory mattresses and kernels and small vibration of mechanical systems (in Russian). M.-L.:, 1950.